

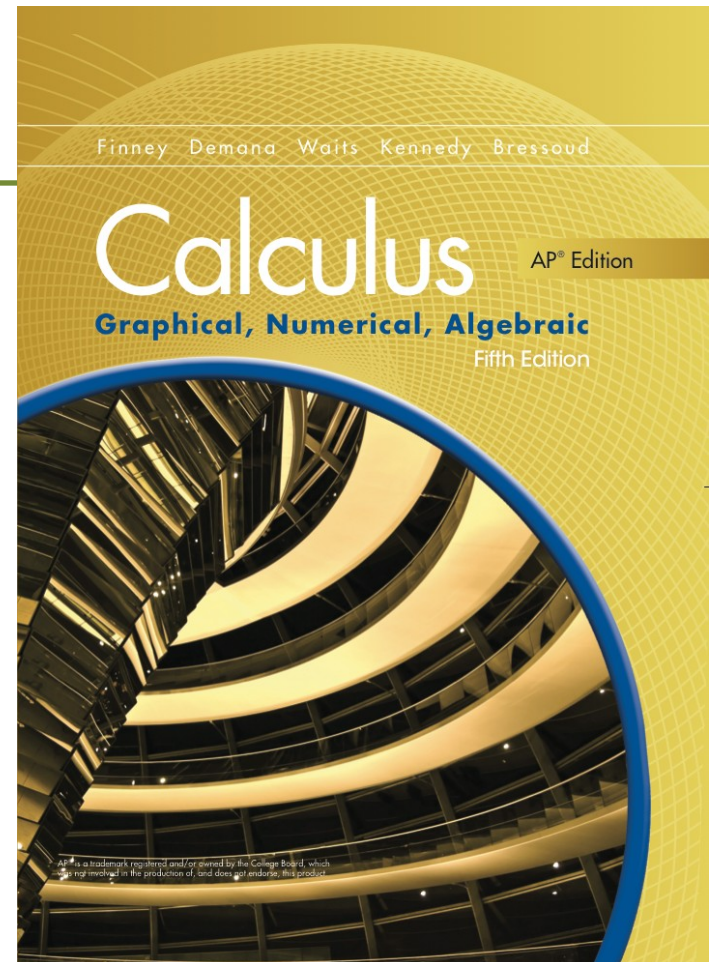
Chapter 11

Parametric, Vector, and Polar Functions



Section 11.1

Parametric Functions



Quick Review

Use algebra or a trig identity to write an equation relating x and y .

1. $x = t + 1$ and $y = 2t + 3$

2. $x = 3t$ and $y = 54t^2 - 3$

3. $x = \sin t$ and $y = \cos t$

4. $x = \sin t \cos t$ and $y = \sin(2t)$

5. $x = \tan \theta$ and $y = \sec \theta$

6. $x = \sin \theta$ and $y = \cos(2\theta)$

Quick Review Solutions

Use algebra or a trig identity to write an equation relating x and y .

1. $x = t + 1$ and $y = 2t + 3$ $y = 2x + 1$

2. $x = 3t$ and $y = 54t^2 - 3$ $y = 6x^2 - 3$

3. $x = \sin t$ and $y = \cos t$ $x^2 + y^2 = 1$

4. $x = \sin t \cos t$ and $y = \sin(2t)$ $y = 2x$

5. $x = \tan \theta$ and $y = \sec \theta$ $y^2 = 1 + x^2$

6. $x = \sin \theta$ and $y = \cos(2\theta)$ $y = 1 - 2x^2$

What you'll learn about

- The slope and concavity of a curve given by parametric functions
- The length of a curve (arc length) given by parametric functions

...and why

Parametric equations enable us to define some interesting and important curves that would be difficult or impossible to define in the form $y=f(x)$.

Example Reviewing Some Parametric Curves

Sketch the parametric curve and eliminate the parameter to find an equation that relates x and y directly.

(a) $x = \cos t$ and $y = \sin t$ for t in the interval $[0, 2\pi)$

(b) $x = 3\cos t$ and $y = 2\sin t$ for t in the interval $[0, 4\pi]$

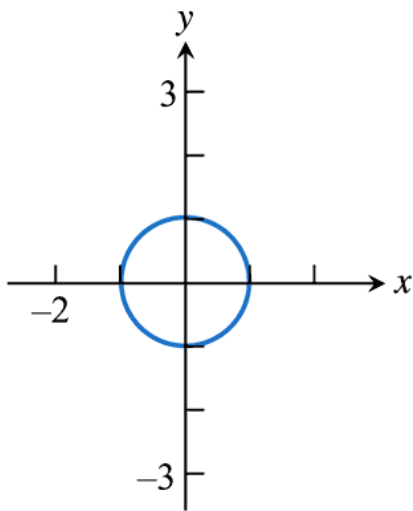
(c) $x = \sqrt{t}$ and $y = t - 2$ for t in the interval $[0, 4]$

(a) Use the identity $\cos^2 t + \sin^2 t = 1$ to find $x^2 + y^2 = 1$.

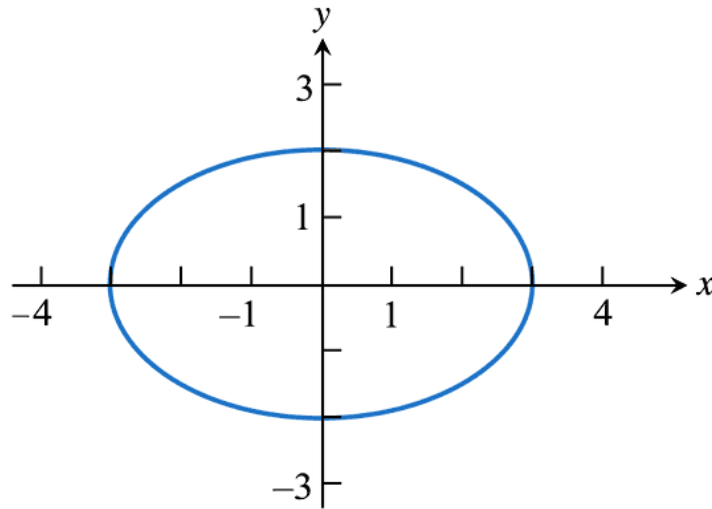
(b) $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$

(c) $y = x^2 - 2$

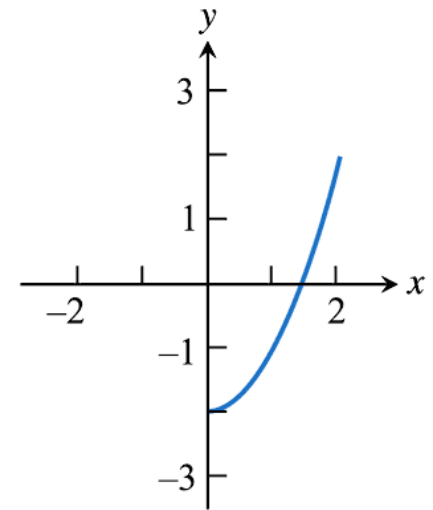
Example Reviewing Some Parametric Curves



(a)



(b)



(c)

Parametric Differentiation Formulas

If x and y are both differentiable functions of t and if $dx/dt \neq 0$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

If $y' = dy/dx$ is also a differentiable function of t , then

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}(y') = \frac{dy'/dt}{dx/dt}.$$

Example Analyzing a Parametric Curve

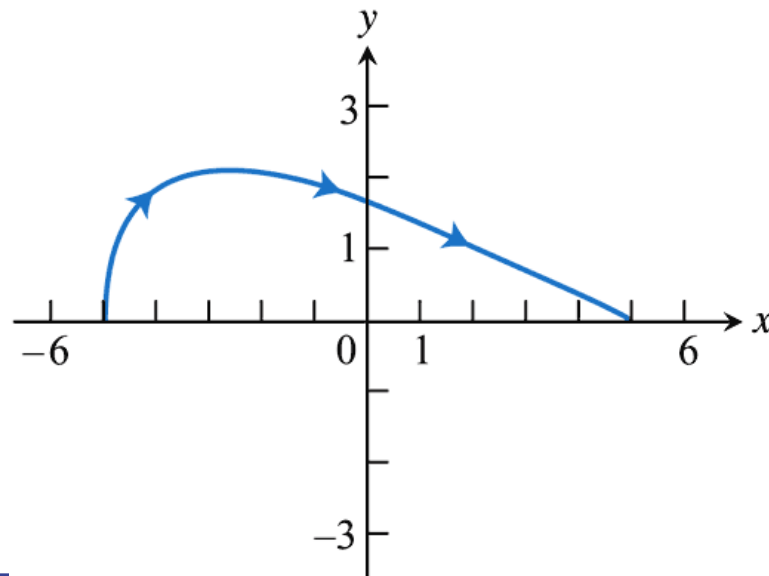
Consider the curve defined parametrically by $x = t^2 - 5$ and $y = 2\sin t$ for $0 \leq t \leq \pi$.

(a) Sketch a graph of the curve in the viewing window $[-7, 7]$ by $[-4, 4]$.

Indicate the direction in which it is traced.

(b) Find the highest point on the curve. Justify your answer.

(c) Find all points of inflection on the curve. Justify your answer.



Example Analyzing a Parametric Curve

(b) Find $\frac{dy}{dt} = 2\cos t$. Since $dy/dt > 0$ for $0 \leq t < \pi/2$ and $dy/dt < 0$ for $\pi/2 < t \leq \pi$, there is a max at $t = \pi/2$. Use this value to find the point $(-2.533, 2)$.

$$(c) \text{ Find } \frac{d^2 y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{(-\sin t)(t) - (1)(\cos t)}{t^2} = -\frac{t\sin t + \cos t}{2t^3}$$

The graph of $y = -\frac{t\sin t + \cos t}{2t^2}$ on the interval $[0, \pi]$ shows a sign change at $t = 2.7983\dots$. Use this value to find the point of inflection at $(2.831, 0.673)$.

Arc Length of a Parametrized Curve

Let L be the length of a parametric curve that is traversed exactly once as t increases from t_1 to t_2 .

If dx/dt and dy/dt are continuous functions of t , then

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} dt.$$

Example Measuring a Parametric Curve

Find the length of the curve defined by $x = \sin t$, $y = \cos t$, $0 \leq t < 2\pi$.

The curve is traced once as t goes from 0 to 2π . Because of the curve's symmetry with respect to the coordinate axes, its length is four times the length of the first quadrant portion.

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\cos^2 t + \sin^2 t} = 1$$

$$L = 4 \int_0^{\pi/2} 1 dt = 4 \left[t \right]_0^{\pi/2} = 4(\pi / 2) = 2\pi. \text{ The length of the curve is } 2\pi.$$

Cycloids

Suppose that a wheel of radius a rolls along a horizontal line without slipping. The path traced by a point P on the wheel's edge is a **cycloid**, where P is originally at the origin.

